Transition behavior in the capacity of correlated-noisy channels in arbitrary dimensions

V. Karimipour [1]
L. Memarzadeh[2],
Department of Physics, Sharif University of Technology,

P.O. Box 11365-9161,
Tehran, Iran

Abstract

We construct a class of quantum channels in arbitrary dimensions for which entanglement improves the performance of the channel. The channels have correlated noise and when the level of correlation passes a critical value we see a sharp transition in the optimal input states (states which minimize the output entropy) from separable to maximally entangled states. We show that for a subclass of channels with some extra conditions, including the examples which we consider, the states which minimize the output entropy are the ones which maximize the mutual information.

PACS Numbers: 03.67.-a, 03.67.Hk

[1] Corresponding author, email:vahid@sharif.edu
[2] email:laleh@mehr.sharif.edu

1 Introduction

One of the basic problems in quantum information theory [1, 2, 3, 4, 5] concerns the issue of additivity of classical capacity of quantum channels. A basic question is whether the use of entangled states as optimal states for encoding classical information can increase the capacity of a channel or not. A proper calculation of the so called Schumacher-Westmoreland-Holevo capacity [3],[4] requires the optimization of mutual information between the input and output of the channel when we encode the information into arbitrary long strings of quantum states (more precisely states in the tensor product of the Hilbert space of one state) and carrying out the limiting procedure $C := \lim_{n\to\infty} C_n$, where

$$C_n := \frac{1}{n} Sup_{\varepsilon} I_n(\varepsilon) \tag{1}$$

is the capacity of the channel when we send $n-$ strings of quantum states into the channel. Here $\varepsilon := \{p_i, \rho_i\}$ is the ensemble of input states,

$$I_n(\varepsilon) := S(\mathcal{E}(\sum_i p_i\rho_i)) - \sum_i p_i S(\mathcal{E}(\rho_i)) \tag{2}$$

is the mutual information between the input and the output, when the channel maps each input state ρ_i into the output state $\mathcal{E}(\rho_i)$, and $S(\rho) \equiv -tr(\rho \log \rho)$ is the von Neumann entropy of a state ρ. We should stress that (1) may not be the proper definition for capacity in a memory channel as formulated in its most general setting in [6], however such a definition seems to be still valid for generic memory channels [6].

Calculation of C is extremely difficult if not impossible. A much simpler problem is to calculate C_2, and to see if entangled states can enhance this kind of limited capacity or not. This problem has been tackled by many authors [7, 8, 9, 10, 11, 12, 13, 14] and there is now strong support for a conjecture that if the noise of the channel is not correlated, i.e for product channels, entangled states have no advantage over separable states for encoding classical information. However when the noise is correlated, several examples have been provided which indicate that entanglement can enhance the mutual information, if the correlation is above a certain critical value. To our knowledge these examples are limited to qubit channels [10],[13], and bosonic Gaussian channels [14].

It is a nontrivial problem to find examples which show such a transition. The difficulty in finding more examples resides in the large number of parameters over which the required optimization should be carried out. In fact one has to propose a channel and a certain type of correlation and only after carrying out the optimization for all values of the noise parameters and correlation values one can see if a critical value of correlation exists above which entangled states are advantageous over separable states. When one goes to higher dimensional states, the number of parameters increases and the problem becomes even more intractable. Therefore it is desirable to have a systematic method for constructing such channels. This is a problem which

we study in this paper.

We will find a general class of channels for which there is a sharp transition, reminiscent of phase transitions, at which the optimal state changes abruptly from a separable state to a maximally entangled state. This transition always happens regardless of the value of noise parameters. Therefore it is remarkable that the struggle for optimality is between these two extremes of entanglement and not between other intermediate values.

The structure of this paper is as follows: In section (2) we first use the minimum output entropy as a parameter characterizing the performance of the channel [15]. Since an input state usually becomes entangled with the environment at the output and the state of the environment is not accessible, minimum entropy at the output means that minimum information has leaked to the environment.

We then provide a general class of channels which are guaranteed to show such transitions in their performance. In section (3) we show that for a subclass of these channels, i.e. those for which the Kraus operators form an irreducible representation of a group and commute modulo a phase with each other, the minimization of output entropy is equivalent to the maximization of the mutual information, hence we show that in the models which we study, which are among the above subclass, it is really the mutual information that behaves non-analytically. In section (4) we study a Pauli channel for qubits with only bit-flip and phase flip errors and show that it shows such a transition. Finally in section (5) we go to arbitrary dimensions and study a subclass of generalized Pauli channels having a symmetry. For this subclass, still satisfying the group representation property, we do the main part of the analysis analytically and only at the end use numerical calculations. Figures (1) and (2) show some of our results.

We should be clear that these and previous similar results on enhancement of mutual information do not directly address the problem of additivity of entropy, since this property deals with sending entangled states over a product channel and not a correlated one.

2 Correlated channels with entanglement-enhanced performance

In arbitrary dimensions consider the following two channels, each of them acts on a single quantum state:

$$\begin{aligned}\Phi(\rho) &= \sum_\alpha p_\alpha U_\alpha \rho {U_\alpha}^\dagger,\\ \Phi^*(\rho) &= \sum_\alpha p_\alpha U^*_\alpha \rho U^{*\dagger}_\alpha. \end{aligned} \tag{3}$$

We take the Kraus operators [16] U_α, to be unitary operators and $\sum_\alpha p_\alpha = 1$. We now consider the channel $\mathcal{E}$ acting on two states as follows:

$$\mathcal{E}(\rho) := (1-\mu)(\Phi \otimes \Phi^*)(\rho) + \mu \Phi^c(\rho), \tag{4}$$

where

$$\Phi^c(\rho) = \sum_\alpha p_\alpha (U_\alpha \otimes U_\alpha^*)\rho(U_\alpha \otimes U_\alpha^*)^\dagger. \tag{5}$$

This type of correlation is inspired by the work of [10] who first proposed it for the Pauli channels. The basic difference however is that our product channel (for $\mu = 0$) is $\Phi \otimes \Phi^*$ rather than $\Phi \otimes \Phi$, and for $\mu = 1$ error operators are of the form $U_\alpha \otimes U_\alpha^*$ instead of $U_\alpha \otimes U_\alpha$. One can not however attach the same physical interpretation as authors of [10] did for this kind of channel, that is one can not interpret this channel as two consecutive uses of the channel $\Phi(\rho)$ in equation (3). However this practical problem is of minor importance, as long as we are interested in the non-analytical behavior of this channel.

To study the performance of this channel we use minimum output entropy as explained in the introduction. That is we search for input states which give the minimum entropy among all output states.

We know that when $\mu = 0$, the channel (4) is a product channel, and for these channels, there is strong analytical and numerical support [9] that the optimal input states of the two channels when multiplied by each other give the best input state of the product channel in the sense that it gives the minimum output entropy. On the other hand we know that when $\mu = 1$, the maximally entangled states pass through the channel without any distortion. This is due to the easily proved identity that for any such state, namely for any state of the form

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i, i\rangle, \tag{6}$$

and for any unitary operator U

$$U \otimes U^* |\psi\rangle = |\psi\rangle. \tag{7}$$

Thus we have $\Phi^c(|\psi\rangle) = |\psi\rangle$ and $S(\Phi^c(|\psi\rangle)) = 0$.

However this does not by itself prove that at $\mu = 1$ separable states may not be optimal, since they may have vanishing output entropy at $\mu = 1$, too. If we can prove that at $\mu = 1$, no separable state gives a vanishing output entropy, then we conclude that somewhere in the interval $[0\ ,\ 1]$, a transition occurs in the behavior of the channel. This by itself does not imply that this transition should be abrupt, however our examples clearly indicate such an abrupt transition.

In order to present our next result, we need a definition. Let us choose an arbitrary Kraus operator say U_{α_0} from the set of U_α's describing the channel. For each α we define the set of states which are invariant modulo a phase under the action of $U_\alpha^{-1}U_{\alpha_0}$, that is

$$I_\alpha := \{|\psi\rangle \in \mathcal{H} \quad | \quad U_\alpha^{-1}U_{\alpha_0}|\psi\rangle = e^{ic_\alpha}|\psi\rangle\}. \tag{8}$$

Then the condition under which a transition occurs is simply given in the following

Theorem: If $\bigcap_\alpha I_\alpha = \emptyset$ then there is a transition in the optimal input states, which improve the channel performance, from separable to maximally entangled states as we increase the level of correlation.

Proof: We want to show that under this condition no separable input state leads to an output state of vanishing entropy.

Let ρ^* be a general separable state. By definition it can be written as a convex combination of pure product states, namely $\rho^* = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)}$, where $\rho_i^{(1)}$ and $\rho_i^{(2)}$ are pure. If $S(\Phi^c(\rho^*)) = 0$, then we find from concavity of S that

$$S(\Phi^c(\rho^*)) = S(\Phi^c(\sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)})) = S(\sum_i p_i \Phi^c(\rho_i^{(1)} \otimes \rho_i^{(2)})) \geq \sum_i p_i S(\Phi^c(\rho_i^{(1)} \otimes \rho_i^{(2)})). \tag{9}$$

In view of positivity of S this means that for all i, $S(\Phi^c(\rho_i^{(1)} \otimes \rho_i^{(2)})) = 0$. Thus we should consider only separable states of the form $\rho_1 \otimes \rho_2$ where ρ_1 and ρ_2 are both pure.

When $\mu = 1$ such a state transforms to the output state

$$\Phi^c(\rho_1 \otimes \rho_2) = \sum_\alpha p_\alpha (U_\alpha \rho_1 U_\alpha{}^\dagger) \otimes (U_\alpha^* \rho_2 U_\alpha{}^{*\dagger}). \tag{10}$$

Again using concavity of S we have

$$S(\Phi^c(\rho_1 \otimes \rho_2)) \geq \sum_\alpha p_\alpha S((U_\alpha \rho_1 U_\alpha{}^\dagger) \otimes (U_\alpha^* \rho_2 U_\alpha{}^{*\dagger})) = \sum_\alpha p_\alpha (S(\rho_1) + S(\rho_2)) = 0, \tag{11}$$

with equality only if all the states $U_\alpha \rho_1 U_\alpha^\dagger$ and $U^*{}_\alpha \rho_2 U_\alpha^T$ are independent of the index α.

This is possible only if we can find a pure state $|\psi\rangle$ which transforms to the same state $|\phi\rangle$ under the action of all the operators U_α, i.e. if

$$\exists \; |\psi\rangle : \quad U_\alpha|\psi\rangle = e^{ic'_\alpha}|\phi\rangle \qquad \forall \alpha, \tag{12}$$

where c'_α is an arbitrary phase. This last condition is however exactly equivalent to the existence of a state $|\psi\rangle$ which is invariant modulo a phase under all the operators $U_\alpha^{-1}U_{\alpha_0}$ for an arbitrarily chosen α_0, i.e.

$$U_\alpha^{-1}U_{\alpha_0}|\psi\rangle = e^{ic_\alpha}|\psi\rangle. \tag{13}$$

Therefore if $\bigcap_\alpha I_\alpha = \emptyset$, no separable state can achieve zero entropy at the output of the channel and a critical value of μ certainly exists.

Note that this theorem by itself does not imply that such a transition is sharp, however the examples of [10, 13, 14] and those presented here suggest such a conclusion. As an example, we are certain that in two dimensions, a channel with Kraus operators I, X, Z will show a transition while a channel with Kraus operators X, Z will not, a result which we have seen in our numerical searches of optimal states.

3 Correlated channels with entanglement-enhanced capacity

If we impose some extra conditions on the Kraus operators and follow the arguments in [13], we can show that minimum output entropy is equivalent to maximal mutual information in the channel. Thus for these kinds of channels the conditions stated in our theorem also guarantee an enhancement of mutual information.

Following [13] we note that the first term of mutual information (2), is maximized if $\sum_i p_i \mathcal{E}(\rho_i) = \frac{1}{d^2} I$. Therefore an upper bound is found for the mutual information in the form

$$I_2 \leq 2\log_2 d - S(\mathcal{E}(\rho^*)). \tag{14}$$

Let us suppose that the Kraus operators in equation (3) have the following two properties: (a) commute with each other modulo a phase: $U_a U_{a'} = U_{a'} U_a e^{i\phi_{\alpha,\alpha'}}$, and (b) form an irreducible representation of a group .
Using property (a) and defining an equiprobable input ensemble $\{p_{\alpha,\alpha'} = \text{const}, \rho_{\alpha,\alpha'}\}$ in which

$$\rho_{\alpha\alpha'} := (U_\alpha \otimes U^*_{\alpha'})\rho^*(U_\alpha \otimes U^*_{\alpha'})^\dagger$$

it is straightforward to show that for $\mathcal{E}$ defined in equation (4) we have:

$$\mathcal{E}(\rho_{\alpha\alpha'}) = (U_\alpha \otimes U^*_{\alpha'})\mathcal{E}(\rho^*)(U_\alpha \otimes U^*_{\alpha'})^\dagger. \tag{15}$$

This mean that the channel $\mathcal{E}$ in covariant with respect to the Kraus operators U_α. Since entropy is invariant under unitary operations, we conclude that

$$S(\mathcal{E}(\rho_{\alpha\alpha'})) = S(\mathcal{E}(\rho^*)) \tag{16}$$

Therefore if ρ^* minimizes the output entropy all the $\rho_{\alpha\alpha'}$ do that, too.

As a consequence of the second property of the Kraus operators, we find that the state $\sum_{\alpha,\alpha'} p_{\alpha,\alpha'}\mathcal{E}(\rho_{\alpha,\alpha'})$ with all $p_{\alpha,\alpha'}$'s equal, commutes with all the operators $U_\alpha \otimes U_\alpha^\dagger$ and hence by Schur's first lemma it is a multiple of identity,

$$\mathcal{E}(\sum_{\alpha\alpha'} p_{\alpha,\alpha'}\rho_{\alpha\alpha'}) = \frac{1}{d^2} I \tag{17}$$

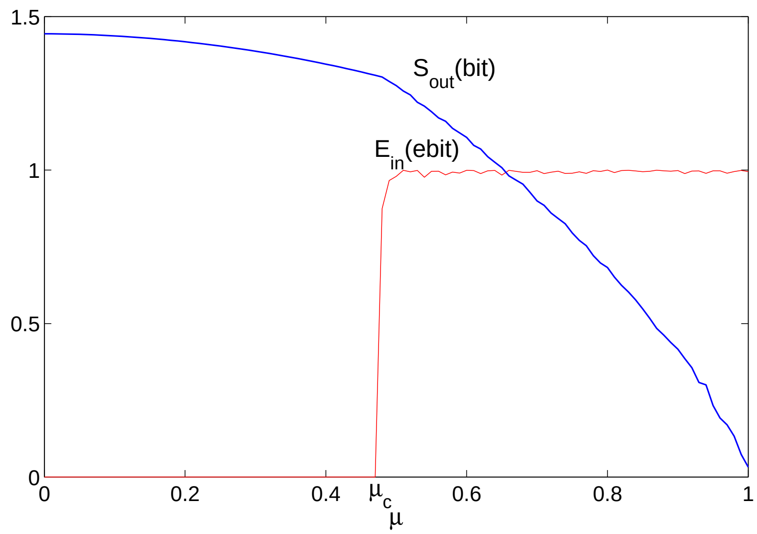

Figure 1: (Color online) Minimum output entropy and entanglement of the related input state as a function of μ, for $p = 0.3$, $q = 0.2$ and $r = 0.5$.

From (2), (16) and (17) we see that the upper bound of equation (14) is attainable if we use $\rho_{\alpha\alpha'}$ with the same probability as the input states. This means that to maximize the mutual information, we only need to find a state ρ^* which minimizes the output entropy.

The examples which we present in the following sections are of this type and hence the transition observed in their behavior as measured in the minimum output entropy is in fact a transition in their mutual information.

4 A qubit channel with correlated noise

In [10] and [13] two examples of qubit channels which have such a critical correlation have been studied. We introduce a third example. Let us take the error operators to be

$$U_1 = I \qquad U_2 = \sigma_x \qquad U_3 = \sigma_z \ , \tag{18}$$

where I is the identity matrix and σ_x and σ_z are the Pauli matrices. These errors happen with probability p, q and r respectively, with $p+q+r=1$. This is a channel with only bit-flip and phase-flip operators.

It is easily verified that this channel satisfies the criteria mentioned in the previous section. We expect this channel to show a transition as we increase the correlation parameter μ. To see this we take a general pure state of two qubits

$$|\psi\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle, \tag{19}$$

and calculate the output state. The eigenvalues of the output density matrix can not be determined analytically and have to be evaluated numerically for fixed correlation parameter μ and error parameters p and q for all states. For each set of these parameters we find the optimal state, i.e. the input state which yields the output state with minimum entropy. The result is shown in figure (1) where we have fixed error parameters to $p = 0.3$ and $q = 0.2$.

It is seen that there is a sharp transition for optimal input state at $\mu_c = 0.47$ from separable states to maximally entangled states.

5 Pauli channels in arbitrary dimensions

In this section we analyze correlated Pauli channels in arbitrary dimensions and study in more detail a 3 dimensional channel, which shows such a critical transition.
In general for a generalized Pauli channel carrying d dimensional states (with basis states $|0\rangle, \cdots |d-1\rangle$), the error operators are the generalized Pauli operators U_{mn} defined as

$$U_{mn}|k\rangle := \xi^{kn}|k+m\rangle, \tag{20}$$

where $\xi := e^{\frac{2\pi i}{d}}$. These operators have well-known properties, including

$$\begin{aligned} U_{mn}^{\dagger} &= \xi^{mn}U_{-m,-n}, \\ U_{kl}U_{mn} &= \xi^{lm-kn}U_{mn}U_{kl}, \\ tr(U_{mn}) &= d\delta_{m,0}\delta_{n,0}, \end{aligned} \tag{21}$$

and satisfy the conditions in section (3). The effect of a Pauli channel on a single qudit is defined as

$$\Phi(\rho) = \sum_{m,n=0}^{d-1} p_{m,n}U_{mn}\rho U_{mn}^{\dagger}, \tag{22}$$

with $\sum_{m,n} p_{m,n} = 1$.

In order to simplify the calculations we can restrict ourselves to a subclass of such channels which have a symmetry.

Let us assume that such a channel has a symmetry of the form

$$\Phi(S_\alpha \rho S_\alpha^{\dagger}) = \Phi(\rho), \tag{23}$$

for α belonging to an index set representing the symmetry group G. Then if ρ^* is an optimal state, $\tilde{\rho} := \frac{1}{|G|}\sum_\alpha S_\alpha \rho^* S_\alpha^{\dagger}$ will also be an optimal state, where $|G|$ is the order of the group. Moreover the state $\tilde{\rho}$ is invariant, that is $S\tilde{\rho}S^{\dagger} = \tilde{\rho} \quad \forall S \in G$. This invariance greatly facilitates our analytical or numerical search for optimal states.
In the Pauli channel let us consider a subclass for which $p_{m,n} = p_m$. This is a generalization of the channel studied in [13]. In this subclass we have the following symmetry

$$\Phi(U_{0k}\rho U_{0k}^{\dagger}) = \Phi(\rho) \qquad \forall k. \tag{24}$$

This symmetry also exists when the channel (22) acts on two qudits in the presence of correlated noise. In that case it takes the form

$$\mathcal{E}(U_{0k} \otimes U^*_{0k})\rho(U_{0k} \otimes U^*_{0k})^\dagger) = \mathcal{E}(\rho). \tag{25}$$

Since $U_{0k} = (U_{01})^k$ this symmetry is generated by only one single element namely U_{01} and we can search for the optimal input state among those which have the following symmetry.

$$(U_{01} \otimes U^*_{01})\rho^*(U_{01} \otimes U^*_{01})^\dagger = \rho^*. \tag{26}$$

A simple calculation in the basis in which U_{01} is diagonal shows that the state $\tilde{\rho}$ is nothing but a convex combination of pure states of the following form

$$|\psi_k\rangle := \sum_{j=0}^{d-1} a_j |j, j-k\rangle, \qquad k = 0, 1, \cdots d-1. \tag{27}$$

Following the reasoning of [13] we can take the optimal state to be a pure state which we take to be of the form $|\psi_0\rangle = \sum_{j=0}^{d-1} a_j |j, j\rangle$, without loss of generality. To find the output state we calculate the correlated and uncorrelated parts of the channel separately. For the correlated part we find

$$\begin{aligned} \Phi^c(|\psi_0\rangle) &= \sum_{m,n,i,j} p_m a_i a_j^* (U_{mn} \otimes U^*_{mn})|i,i\rangle\langle j,j|(U_{mn} \otimes U^*_{mn})^\dagger \\ &= \sum_{m,i,j} d p_m a_i a_j^* |i+m, i+m\rangle\langle j+m, j+m|. \end{aligned} \tag{28}$$

For the uncorrelated part we have

$$(\Phi \otimes \Phi^*)(|\psi_0\rangle) = \sum_{i,j} a_i a_j^* \Phi(|i\rangle\langle j|) \otimes \Phi^*(|i\rangle\langle j|). \tag{29}$$

Since

$$\begin{aligned} \Phi(|i\rangle\langle j|) &= \sum_m p_m U_{mn} |i\rangle\langle j| U^\dagger_{mn} = \sum_{m,n} p_m \xi^{(i-j)n} |i+m\rangle\langle j+m| \\ &= d\delta_{ij} (\sum_m p_m |i+m\rangle\langle i+m|). \end{aligned} \tag{30}$$

we find

$$(\Phi \otimes \Phi^*)(|\psi_0\rangle) = d^2 \sum_{i,m,n} p_m p_n |a_i|^2 |i+m, i+n\rangle\langle i+m, i+n|. \tag{31}$$

As in (4) the complete output of the channel will be

$$\mathcal{E}(|\psi_0\rangle) = (1-\mu)(\Phi \otimes \Phi^*)(|\psi_0\rangle) + \mu \Phi^c(|\psi_0\rangle). \tag{32}$$

One can now determine the entropy of this output state and minimize it with respect to the coefficients a_i to determine the optimal input state and its entanglement.

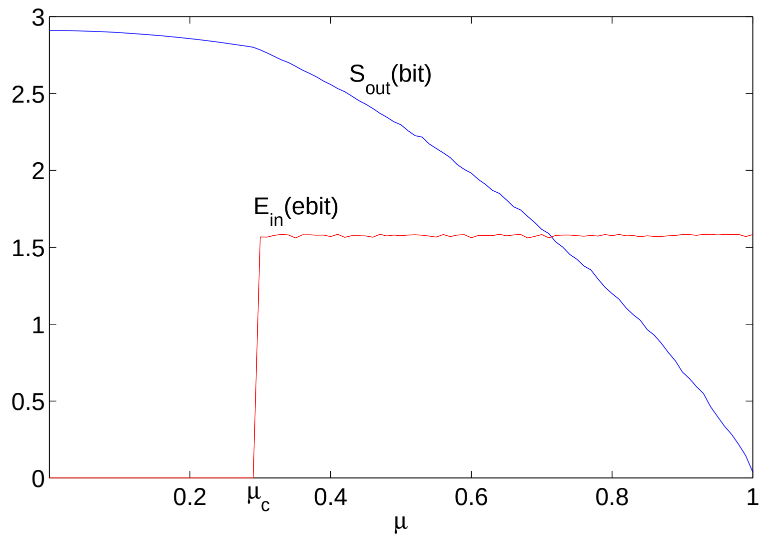

Figure 2: (Color Online) The minimum output entropy and the entanglement of the optimal state as a function of μ for a 3 dimensional symmetric Pauli channel. The critical value of μ is $\mu_c \approx 0.29$ for $p = 0.0800$,$q = 0.1800$ and $r = 0.0733$.

This part of the problem must usually be carried out numerically. We have done this task for 3 level states (qutrits), where the matrix of error parameters take the form

$$p_{m,n} \equiv \begin{pmatrix} p & p & p \\ q & q & q \\ r & r & r \end{pmatrix}. \tag{33}$$

For fixed error parameters, and for variable values of the correlation parameters μ, we have searched numerically among all the states which minimize the output entropy. Figure (2) shows the entropy of the output state when the optimal state is fed into the channel. For each μ the entanglement of the optimal input state is also plotted. It is clearly seen that there is a sharp transition at $\mu_c \approx 0.29$. Below μ_c the optimal state is a separable state and above μ_c it is a maximally entangled state. This plot is typical, changing the error parameters only changes the value of critical correlation μ_c. Note that in calculating the entanglement of the input state we have used logarithms to base 2 so that a maximally entangled state has an entanglement of $\log_2 3$.

The interesting features are that first, the transition is sharp and not smooth and second no matter what the error parameters are, it is the maximally entangled states and not some other states with lower values of entanglement which minimize the output entropy and hence maximize the mutual information. Therefore the transition is governed by a struggle of the two extremes of entanglement.

Before concluding the paper it is instructive to compare the fidelity of the output and input states and also the linearized entropies of the output state for two extreme cases, namely when the input state is completely separable and when it is a maximally entangled state. This will provide us with a very simple way to obtain an estimate of the critical value of correlation.

For the maximally entangled state $\rho^{ME} := |\psi\rangle\langle\psi|$ where $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_i |i,i\rangle$, we put $a_i = \frac{1}{\sqrt{d}}$ in (28) and (31) and find

$$\mathcal{E}(\rho^{ME}) = (1-\mu)d\sum_{m,n} C_{m,n}|m,n\rangle\langle m,n| + \mu\rho^{ME} \tag{34}$$

where

$$C_{mn} = \sum_i p_{m+i}p_{n+i}. \tag{35}$$

For a separable state $\rho^s := |0,0\rangle\langle 0,0|$ we find

$$\mathcal{E}(\rho^s) = (1-\mu)\chi\otimes\chi + \mu d\sum_m p_m|m,m\rangle\langle m,m|, \tag{36}$$

where $\chi = d\sum_m p_m|m\rangle\langle m|$. (Other separable states like $|k,k\rangle$ give different output states but the same output entropy).

For the maximally entangled state the fidelity will be

$$F^{ME} := \langle\psi|\mathcal{E}(\rho^{ME})|\psi\rangle = \mu + (1-\mu)\sum_n C_{nn} = \mu + (1-\mu)d\sum_n p_n^2. \tag{37}$$

and the linearized entropy $R(\rho^{ME}) = 1 - tr(\mathcal{E}^2(\rho^{ME}))$ will be

$$R^{ME} = 1 - [(1-\mu)^2 d^2 \sum_{m,n} C_{mn}^2 + \mu^2 + 2\mu(1-\mu)dC_{00}] \tag{38}$$

while for separable states the corresponding quantities will be

$$F^s = \langle 0,0|\mathcal{E}(\rho^s)|0,0\rangle = (1-\mu)d^2p_0^2 + \mu d p_0. \tag{39}$$

and

$$R^s = 1 - [(1-\mu)^2 d^4 C_{00}^2 + \mu^2 d^2 C_{00} + 2\mu(1-\mu)d^3\sum_m p_m^3]. \tag{40}$$

For the 3 dimensional channel that we have studied with the parameters $p_0 = 0.08$, $p_1 = 0.18$ and $p_2 = 0.073$ we have plotted in figure (3) and (4) the linearized entropies of these two output states and their fidelity with their input states.

We see that the maximally entangled states have a higher fidelity than separable states at the output for all values of μ, however their output linearized entropy becomes less than that of the separable states at almost the same critical value of $\mu_c \approx 0.28$ which we found by considerations of minimum von-Neumann entropy.

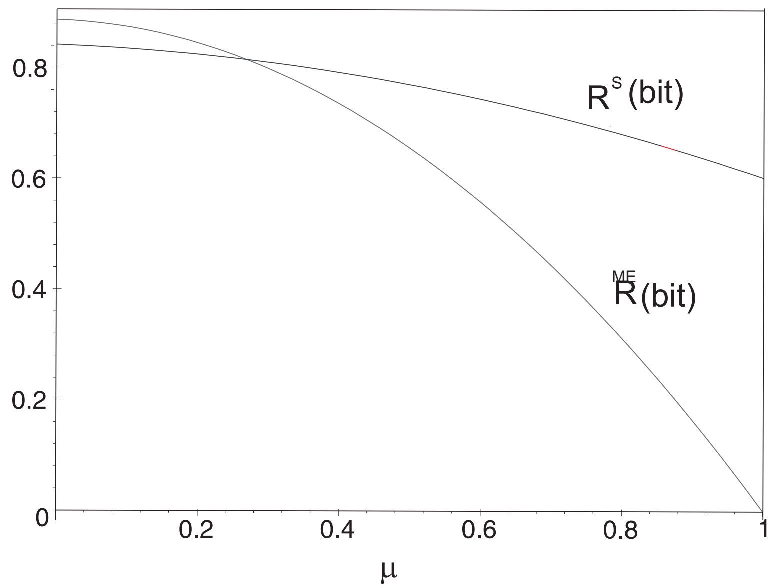

Figure 3: The linearized output entropies for the maximally entangled states (R^{ME}) and separable states (R^s) as a function of μ.

Therefore it may be possible to analyze more complicated channels, those without symmetry and in arbitrary dimensions, in a much simpler way, i.e. by searching for optimal states either numerically or analytically, according to the minimality of their linearized and not von-Neumann entropy.

6 discussion

We have provided conditions under which a channel with correlated noise shows a sharp transition in the form of its optimal states, as the level of correlation passes a critical value. The interesting point is that the transition occurs from completely separable states to maximally entangled states and not states with some intermediate value of entanglement, depending on the values of error parameters. This phenomenon is reminiscent of phase transitions. In the same way that in phase transitions there is a struggle between order and disorder, i.e. between energy and entropy, here there is a struggle between maximal entanglement and complete separability. One is tempted to link this to a symmetry breaking phenomenon. In ferromagnetic phase transitions we know that the free energy changes its shape when we lower the temperature below the critical temperature, and a unique minimum (with zero magnetization) bifurcates to a a manifold of minima (with non-zero magnetization). Is there a similar function here defined on the space of states or their entanglement which undergo a similar change when we increase the level of correlation? We think that this question deserves much further investigation.

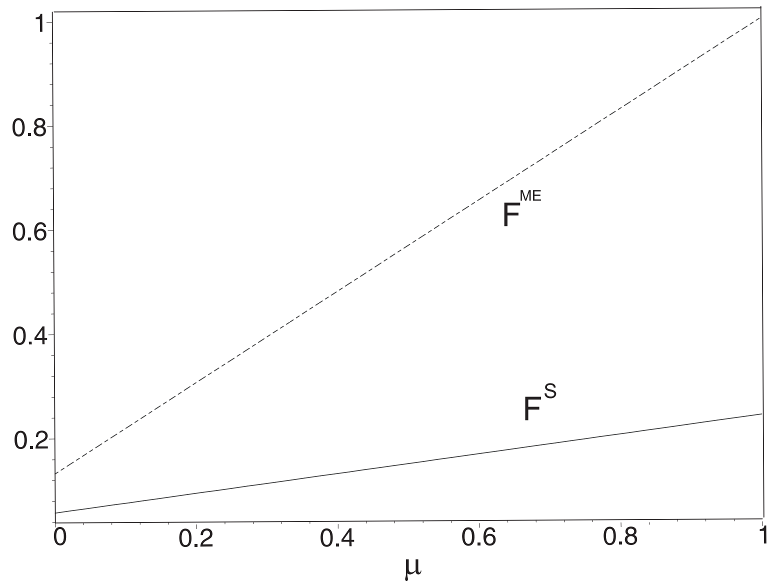

Figure 4: The fidelity of the output and input states for the maximally entangled states (F^{ME}) and separable states (F^s), as a function of μ. The parameter μ and the fidelities are dimensionless.

7 Acknowledgement

We would like to thank the members of the Quantum information group of Sharif University for very valuable comments. Our special thanks go to M. Dehghan Niri for providing generous help with some needed softwares.

References

[1] M. A. Nielsen, and I. L. Chuang;*Quantum computation and quantum information*,Cambridge University Press, Cambridge, 2000.

[2] C.Macchiavello, G.M.Palma and A.Zeilinger, *Quantum computation and quantum information theory*, World Scientific, Singapore, 2001.

[3] A. S. Holevo, IEEE Trans. Info. Theory 44, 269-273 (1998).

[4] B. Schumacher and M. D. Westmoreland, Phys.Rev. A **56**, 131-138 (1997).

[5] P. W. Shor, quant-ph/0305035.

[6] D. Kretschmann, R. F. Werner, quant-ph/0502106

[7] D. Bruß, L. Faoro, C. Macchiavello, and G.M. Palma, J. Mod. Opt. **47**, 325, 2000.

[8] C. King and M.B. Ruskai, IEEE Trans. Inf. Theory **47**,192(2001).

[9] C. King, quant-ph/0103156

[10] C. Macchiavello, and G.M. Palma, Phys. Rev. A **65**, 050301(R)(2002).

[11] K. Matsumoto, T. Shimono and A. Winter, Comm. Math. Phys. 246(3):427-442,(2004).

[12] K. M. R. Audenaert and S. L. Braunstein, Comm. Math. Phys. (246) No 3, 443-452, (2004)

[13] Macchiavello, G.M. Palma, S. Virmani, Phys. Rev. A **69**, 010303, (2004).

[14] N.J. Cerf, J. Clavareau, C. Macchiavello. and J. Roland, Phys. Rev. A **72**, 042330 (2005).

[15] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, quant-ph/0404005

[16] K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* Springer, Berlin, 1983.